Demonstration of Riemann Hypothesis

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June 12, 2014

Abstract

We define an infinite summation which is proportional to the reverse of Riemann Zeta function $\zeta(s)$. Then we demonstrate that such function can have singularities only for $Re\ s = 1/n$ with $n \in \mathbb{N} \setminus 0$. Finally, using the functional equation, we reduce these possibilities to the only $Re\ s = 1/2$.

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1 Target

Riemann hypothesis, proposed by Bernhard Riemann in the 1859, is a conjecture that regards an apparently simple function of complex variable $s$. Such function, called *Riemann Zeta Function*, is defined for $\text{Re } s > 1$ via the following summation

$$
\zeta(s) = \sum_{n=1}^{+\infty} n^{-s}
$$

For every integer $n$, an unique decomposition as product of (powers of) prime numbers exists. In this way

$$
\zeta(s) = \prod_{p=2}^{+\infty} \sum_{j=0}^{+\infty} p^{-sj} = \prod_{p=2}^{+\infty} \frac{1}{1 - p^{-s}}
$$

where we have used the summation rule for the geometric series

$$
\sum_{j=1}^{+\infty} w^j = \frac{1}{1 - w}
$$

for $|w| < 1$. *Riemann Zeta function* can’t have zeros in the convergence area, because no term in the product can be equal to zero. Nevertheless an holomorphic extension of $\zeta(s)$ can be defined over the entire complex plane $\mathbb{C}$, with the exception of $s = 1$. Such extension has infinite zeros corresponding to all negative even integers; that is, $\zeta(s) = 0$ when $s$ is one of $-2, -4, -6, \ldots$. These are the so-called “trivial zeros”.

Again for the holomorphic extension, for $s \neq 1$ we can prove the functional equation\(^1\)

$$
\zeta(s) = 2^s \pi^{s-1} \sin \left( \frac{\pi s}{2} \right) \Gamma(1 - s) \zeta(1 - s)
$$

\(^1\)The first proof was given by Bernhard Riemann in the 10-page paper *Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse* (usual English translation: *On the Number of Primes Less Than a Given Magnitude*) published in the November 1859 edition by Monatsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin.
Leaving out the zeros of \( \sin \left( \frac{\pi s}{2} \right) \) which aren’t poles of \( \Gamma(1 - s) \), i.e. for \( s = -2n, n \in \mathbb{N} \), any other zero \( s_0 \) must have a “mate” zero \( s'_0 = 1 - s_0 \). Because there are no zeros for \( Re s > 1 \), functional equation implies that there are no zeros also for \( Re s < 0 \) (except for \( s = -2n \)). Other works have excluded also the presence of zeros for \( Re s = 0 \) and \( Re s = 1 \). As consequence, all the non trivial zeros of \( \zeta(s) \) stay in the “critical strip” \( 0 < Re s < 1 \).

Riemann hypothesis conjectures that all the non-trivial zeros have real part equal to \( \frac{1}{2} \). This is what we aim to demonstrate in the following.

## 2 Strategy

We start from the definition of \( \zeta(s) \) as an infinite product of terms, one term for every prime number \( p \), running from 2 to \(+\infty\):

\[
\zeta(s) = \prod_{p=2}^{+\infty} \frac{1}{1 - p^{-s}} \tag{1}
\]

The product converges for \( Re s > 1 \). From here we fix \( s = x + iy \) with \( s \in \mathbb{C} \) and \( x, y \in \mathbb{R} \). So the convergence condition is \( x > 1 \). Out of convergence area, \( \zeta(s) \) is defined via holomorphic extension.

The derivative of a single term in the product (1) gives

\[
\frac{d}{ds} \frac{1}{1 - p^{-s}} = -\frac{1}{(1 - p^{-s})^2} \log p \frac{p^{-s}}{p^s - 1}
\]

Hence the derivative \( \zeta'(s) \) of \( \zeta(s) \) results

\[
\zeta'(s) = \frac{d}{ds} \zeta(s) = -\prod_{p=2}^{+\infty} \frac{1}{1 - p^{-s}} \cdot \sum_{p'=2}^{+\infty} \frac{\log p'}{p'^s - 1}
\]

and then

\[
\frac{\zeta'(s)}{\zeta(s)} = -\sum_{p=2}^{+\infty} \frac{\log p}{p^s - 1} \tag{2}
\]
For briefness we set \( C(s) = \frac{\zeta'(s)}{\zeta(s)} \). The sum (2) converges for \( x > 1 \). Otherwise we define \( C(s) \) as the holomorphic extension of (2), recognizing it by the label “H.e.”:

\[
C(s) = -H.e. \sum_{n=1}^\infty \frac{\log p_n}{p_n^s - 1}
\]

The uniqueness of the holomorphic extension ensures that \( C(s) = \frac{\zeta'(s)}{\zeta(s)} \) also for \( x \leq 1 \). At this point we rely upon the following evidence:

**Any singularity of** \( C(s) \) **corresponds to a zero of** \( \zeta(s) \) **and/or to a singularity of** \( \zeta'(s) \).

We can exclude that a zero of \( \zeta'(s) \) hides a zero of \( \zeta(s) \): in fact, for any holomorphic function \( f(s) \), if a point \( s_0 \) exists which is a zero both for \( f(s) \) and \( f'(s) \), we have surely

\[
\frac{f(s_0)}{f'(s_0)} = \lim_{s \to s_0} \sum_{k=1}^\infty \frac{f_k(s-s_0)^k}{\sum_{k=1}^\infty k f_k(s-s_0)^{k-1}} = \lim_{s \to s_0} \frac{f_k(s-s_0)^k}{k f_k(s-s_0)^{k-1}} = \lim_{s \to s_0} \frac{s-s_0}{k} = 0
\]

where \( \hat{k} \) is the minor \( k \) for which a Taylor coefficient \( f_k \) is \( \neq 0 \). Hence the zeros of \( \zeta(s) \) are singularities for \( C(s) \) also when they are zeros of both \( \zeta(s) \) and \( \zeta'(s) \).

For this reason we’ll proceed with finding an ensemble of points which includes all the singularities of \( C(s) \) and so, among them, all the zeros of \( \zeta(s) \).

### 3 Application of the Euler-MacLaurin formula

We move from prime to integer numbers:

\[
C(s) = -\sum_{n=1}^\infty \frac{\log p_n}{p_n^s - 1}
\]  \( (3) \)
where $p_n$ is the $n$-th prime number. For $s \neq 0$, no partial sum

$$C_1^M(s) = -\sum_{n=1}^{M} \frac{\log p_n}{p_n^s - 1}$$

with $M < +\infty$ has singularities. Hence, any function

$$C_M(s) = -\sum_{n=M}^{+\infty} \frac{\log p_n}{p_n^s - 1}$$

has the same singularities of $C(s)$ for $s \neq 0$. This permit us to work with $C_M(s)$ in place of $C(s)$, in such a way to exploit the freedom in choosing $M$. Consider now the Euler-MacLaurin theorem at leading order:

$$\left| \sum_{l=M}^{N} f(l, s) - \int_{M}^{N} f(w, s) dw \right| \leq F^N(s) \in \mathbb{R}^+$$

$$F^N(s) = \int_{M}^{N} dw \left| \frac{d}{dw} f(w, s) \right|$$

for some $f(w, a)$ analytic in $w \in \mathbb{R}$ and holomorphic in $s \in \mathbb{C}$ except at most for isolated points, provided that in such points we have $w \notin \mathbb{N}$.

Consider now $\lim_{N \to \infty} F^N(s) \overset{!}{=} F(s)$. If $F(s)$ exists (is finite) in a open set $x > A$ (with $A$ some real number) except at most for isolated points, then the Dominated Convergence Theorem ensures (for $x > A$) that

$$\lim_{N \to \infty} \left| \sum_{l=M}^{N} f(l, s) - \int_{M}^{N} f(w, s) dw \right| \leq F(s) \quad (4)$$

Now it is possible that the sum and the integral in the left side converge only for $x > B$ with $B > A$. For $A < x \leq B$ we can separate the holomorphic extension of
the summation (H.e.) from its divergent piece (D.p.):

\[ \sum_{l=M}^{\infty} f(l, s) = H.e. \sum_{l=M}^{\infty} f(l, s) + D.p. \sum_{l=M}^{\infty} f(l, s) \]

We can do the same for the integral:

\[ \int_{M}^{\infty} f(w, s)dw = H.e. \int_{M}^{\infty} f(w, s)dw + D.p. \int_{M}^{\infty} f(w, s)dw \]

Inserting them in (4) we obtain:

\[ \left| H.e. \sum_{l=M}^{\infty} f(l, s) + D.p. \sum_{l=M}^{\infty} f(l, s) - H.e. \int_{M}^{\infty} f(w, s)dw - D.p. \int_{M}^{\infty} f(w, s)dw \right| \leq F(s) \]

Being \( F(s) \) finite (except for isolated points), it has to be true

\[ D.p. \sum_{l=M}^{\infty} f(l, s) = D.p. \int_{M}^{\infty} f(w, s)dw \]

and so

\[ \left| H.e. \sum_{l=M}^{\infty} f(l, s) - H.e. \int_{M}^{\infty} f(w, s)dw \right| \leq F(s) \]

for \( A < x \leq B \). Hence we can apply the Euler-Maclaurin formula not only for a comparison between sum and integral, but also between their holomorphic extensions, at least until \( F(s) \) is finite. Our case is

\[ \left| \sum_{n=M}^{\infty} \frac{\log p_n}{p_n^s - 1} - \int_{M}^{\infty} dt \frac{\log p(t)}{p(t)^s - 1} \right| \leq F(s) \]

Here \( p(t) \) is any analytic function which posses analytic inverse \( t(p) \) and satisfies
\( p(n) = p_n \). Moreover

\[
\left| \sum - \int \right| \leq \int_M^\infty dt \left| \frac{d}{dt} \log p(t) \right| \left| \frac{dp(t)}{dt} \left[ \frac{1}{p(t)(p(t)^s - 1)} - \frac{sp(t)^{s-1} \log p(t)}{(p(t)^s - 1)^2} \right] \right| \leq \int_M^\infty dt \left| \frac{dp(t)}{dt} \left[ \frac{1}{p(t)(p(t)^s - 1)} - \frac{sp(t)^{s-1} \log p(t)}{(p(t)^s - 1)^2} \right] \right| (5)
\]

The right side converges for \( x > 0 \) with the exceptions of isolated points.

The function \( t(p_n) \) returns the cardinality \( n \) of the prime number \( p_n \). This is equivalent to return how many prime numbers exist that are less than or equal to \( p_n \), i.e. \( t(p) = \pi(p) \), where \( \pi(p) \) is the prime-counting function. A holomorphic definition was given by Riesel, Edwards and Derbyshire\(^2\):

\[
\pi(p) = \lim_{T \to +\infty} \frac{1}{2\pi i} \int_{2-\i T}^{2+\i T} \frac{p^s}{s} \log \zeta(s) ds
\]

\[
\pi'(p) = \lim_{T \to +\infty} \frac{1}{2\pi i} \int_{2-\i T}^{2+\i T} p^{s-1} \log \zeta(s) ds
\]

For \( p \to +\infty \) we have

\[
\pi(p) \sim \frac{p}{\log p}
\]

\[
\pi'(p) = \frac{d\pi(p)}{dp} \sim \frac{1}{\log p} \left( 1 - \frac{1}{\log p} \right) \sim \frac{1}{\log p}
\]

Before going any further, we prove that (the holomorphic extension of) the integral in \( | \sum - \int | \),

\[
\text{H.e.} \int_{p_M}^{+\infty} dp \frac{dp(t)}{dp} \log p \frac{p^s - 1}{p^s - 1} = \text{H.e.} \int_{p_M}^{+\infty} dp \pi'(p) \log p \frac{p^s - 1}{p^s - 1},
\]  

has no singularities. We introduce first a redefinition of the same integral which permits to calculate it also for \( x \leq 1 \). This is

$$
\frac{1}{2i \sin(\pi s)} \int_{C_s} dp \, \pi'(p) \frac{\log p \, (-p)^{-s}}{1 - p^{-s}},
$$

which is equivalent to (6) for \( x > 1 \), so giving the relative holomorphic extension.

The Cauchy principle ensures that integral maintains the same value for every choice of \( \varepsilon \) (until the path doesn’t touch any pole of the integrated function); for demonstration we find useful to work in the limit \( \varepsilon \to 0^+ \).

The contribute at \( \infty \) is null for \( x > 1 \). This is unique zone of interest to check the equivalence between (6) and (7); so, at this aim, we can forget about it.

\[
\lim_{|p| \to +\infty} \pi'(p) \frac{\log p \, (-p)^{-s}}{1 - p^{-s}} = 0 \quad \text{for} \quad x > 1
\]

Having said this, we proceed by calculating separately the contributes to the integral along \( \Gamma_1 \), \( \Gamma_3 \) and \( \Gamma_2 \):

\[
\int_{\Gamma_1} [ ] = \lim_{\varepsilon \to 0^+} \int_{pM}^{+\infty} dt \, \pi'(t + i\varepsilon) \frac{\log(t + i\varepsilon) e^{-s \log(t + i\varepsilon) + i\pi s}}{1 - e^{-s \log(t + i\varepsilon)}}
\]

\[
= e^{i\pi s} \int_{pM}^{+\infty} dp \, \pi'(p) \frac{\log p \, p^{-s}}{1 - p^{-s}}
\]

\[
= e^{i\pi s} \int_{pM}^{+\infty} dp \, \pi'(p) \frac{\log p}{p^s - 1}
\]

\[
\int_{\Gamma_3} [ ] = \lim_{\varepsilon \to 0^+} \int_{+\infty}^{pM} dt \, \pi'(t - i\varepsilon) \frac{\log(t - i\varepsilon) e^{-s \log(t - i\varepsilon) - i\pi s}}{1 - e^{-s \log(t - i\varepsilon)}}
\]

\[
= -e^{-i\pi s} \int_{pM}^{+\infty} dp \, \pi'(p) \frac{\log p \, p^{-s}}{1 - p^{-s}}
\]

\[
= -e^{-i\pi s} \int_{pM}^{+\infty} dp \, \pi'(p) \frac{\log p}{p^s - 1}
\]
For the integral in $\Gamma_2$, we pose $p = \varepsilon e^{i\theta} + p_M$, $dp = i\varepsilon e^{i\theta}d\theta$,

$$
\int_{\Gamma_2} \frac{p'p\log \left( p^{1-s} \right)}{1 - \left( \varepsilon e^{i\theta} + p_M \right)^{-s}} = \lim_{\varepsilon \to 0} \int_{-3\pi/4}^{3\pi/4} d\theta \frac{\pi'(\varepsilon e^{i\theta} + p_M) \log \left( \varepsilon e^{i\theta} + p_M \right) - s e^{i\pi s} \varepsilon e^{i\theta}}{1 - \left( \varepsilon e^{i\theta} + p_M \right)^{-s}}
$$

$$
= \lim_{\varepsilon \to 0} i\varepsilon \int_{-3\pi/4}^{3\pi/4} d\theta \frac{\pi'(p_M) \log \left( p_M \right) - s e^{i\pi s} \varepsilon e^{i\theta}}{p_M - 1} \int_{-3\pi/4}^{3\pi/4} d\theta \varepsilon e^{i\theta}
$$

$$
= \lim_{\varepsilon \to 0} \varepsilon \int_{-3\pi/4}^{3\pi/4} d\theta \frac{\pi'(p_M) \log \left( p_M \right) e^{i\pi s}}{p_M - 1} \left[ e^{i\pi s} - e^{-i\pi s} \right]
$$

$$
= \lim_{\varepsilon \to 0} i\varepsilon \sqrt{2} \int_{-3\pi/4}^{3\pi/4} d\theta \frac{\pi'(p_M) \log \left( p_M \right) e^{i\pi s}}{p_M - 1} = 0
$$

In the end:

$$
\frac{1}{2i \sin(\pi s)} \int_{C_\varepsilon} dp' \frac{\log p \left( -p \right)^{-s}}{1 - p^{-s}} = \frac{1}{2i \sin(\pi s)} \int_{\Gamma_1} + \int_{\Gamma_2} + \int_{\Gamma_3}
$$

$$
= \frac{1}{2i \sin(\pi s)} \left( e^{i\pi s} - e^{-i\pi s} \right) \int_{p_M}^{+\infty} dp' \frac{\log p}{p^{s} - 1}
$$

$$
= \frac{1}{2i \sin(\pi s)} 2i \sin(\pi s) \int_{p_M}^{+\infty} dp' \frac{\log p}{p^{s} - 1}
$$

$$
= \int_{p_M}^{+\infty} dp' \frac{\log p}{p^{s} - 1}
$$

CVD. At this stage we can doubtless affirm that (7) is the holomorphic extension of (6). Using the Cauchy’s integral theorem, the integral in (7) can be transformed into a sum over the minimal circulations around all the poles. These sit at $p = e^{2\pi i k/s}, k \in \mathbb{Z}$. A single term is the following:

$$
\sin(\pi s) \int_{C_{\varepsilon}}^{(k)} dp' \frac{\log p}{p^{s} - 1} = \lim_{\varepsilon \to 0^+} \frac{i\varepsilon}{2i} \int_{0}^{2\pi} e^{i\theta} d\theta \frac{\pi' \left( e^{2\pi i k/s} + \varepsilon e^{i\theta} \right) \log \left( \frac{e^{2\pi i k/s} + \varepsilon e^{i\theta}}{e^{2\pi i k/s} + \varepsilon e^{i\theta}} \right) - 1}{e^{2\pi i k/s} + \varepsilon e^{i\theta}}
$$

$$
= \lim_{\varepsilon \to 0^+} \varepsilon \int_{0}^{2\pi} e^{i\theta} d\theta \frac{\pi' \left( e^{2\pi i k/s} + \varepsilon e^{i\theta} \right) \log \left( 1 + \varepsilon e^{i\theta - 2\pi i k/s} + \frac{2\pi k}{s} \right)}{e^{2\pi i k/s} \left( 1 + \varepsilon e^{i\theta - 2\pi i k/s} \right) - 1}
$$

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\[ \lim_{\varepsilon \to 0} \frac{\varepsilon}{2} \int_0^{2\pi} e^{i\theta} d\theta \pi'(e^{2\pi ik s} + \varepsilon e^{i\theta}) \log(1 + \varepsilon e^{i\theta} - \frac{2\pi k}{s}) e^{i\pi s} = \]

\[ = \lim_{\varepsilon \to 0} \frac{\varepsilon}{2} \int_0^{2\pi} e^{i\theta} d\theta \pi'(e^{2\pi ik s} + \varepsilon e^{i\theta}) \varepsilon e^{i\theta} - \frac{2\pi k}{s} e^{i\pi s} = \]

\[ = \lim_{\varepsilon \to 0^+} \frac{1}{2} \int_0^{2\pi} e^{i\theta} d\theta \pi'(e^{2\pi ik s}) \varepsilon e^{i\theta} - \frac{2\pi k}{s} e^{i\pi s} = \]

\[ = \frac{\pi k}{s^2} \pi'(e^{2\pi ik s}) e^{\frac{2\pi i k}{s} + i\pi s} \int_0^{2\pi} d\theta = \]

\[ = \frac{2\pi^2 k}{s^2} \pi'(e^{2\pi i k s}) e^{\frac{2\pi i k}{s} + i\pi s} \]

By summing over all the poles:

\[ \sin(\pi s) H.e. \int_{p_M}^{+\infty} dp \pi'(p) \frac{\log p}{p^s - 1} = e^{i\pi s} \sum_{k=-\infty}^{+\infty} \frac{2\pi^2 k}{s^2} \pi'(e^{\frac{2\pi i k}{s}}) e^{\frac{2\pi i k}{s}} \]

\[ = e^{i\pi s} \sum_{k=-\infty}^{+\infty} \frac{2\pi^2 k}{s^2} \pi'(e^{\frac{2\pi i k}{s}}) e^{\frac{2\pi i k}{s} - i|w|^2} \]

\[ = \frac{\pi}{s^2} e^{i\pi s} \int_{-\infty}^{+\infty} \log \zeta(2 + i w) dw \sum_{k=-\infty}^{+\infty} k e^{\frac{2\pi i k (2 + i w)}{|w|^2}} e^{\frac{2\pi i k}{|w|^2} - i|w|^2} \]

\[ = \frac{\pi}{s^2} e^{i\pi s} \int_{-\infty}^{+\infty} \log \zeta(2 + i w) dw \sum_{k=1}^{+\infty} k \left( e^{\frac{2\pi i k (2 + i w)}{|w|^2}} - e^{\frac{2\pi i k}{|w|^2}} - \frac{2\pi i k (2 + i w)}{|w|^2} \right) \]

Use now the summation rule

\[ \sum_{k=1}^{+\infty} k e^{ka} = \frac{d}{da} \sum_{k=1}^{+\infty} e^{ka} = \frac{d}{da} \left[ \frac{1}{1 - e^a} - 1 \right] = \frac{e^a}{(1 - e^a)^2} = \frac{1}{(e^{-a/2} - e^{a/2})^2} \]

The last term in the right is the correct value of summation only for Re a < 0. Nevertheless, when we climb over the line Re a = 0, it gives the corresponding holomorphic extension. Hence we can use the rule without care of convergence.
criterium:

\[ = \frac{\pi}{s^2} e^{i\pi s} \int_{-\infty}^{+\infty} \log \zeta(2 + iw) \, dw \quad \frac{1}{e^{\pi i (2 + iw) s^*} - e^{-\pi i (2 + iw) s^*}} \left| s \right|^2 \]

For \( w \to +\infty \) the integrand goes like

\[ \sim \frac{\pi}{s^2} e^{i\pi s} \log \zeta(2 + iw) e^{-\frac{2w^* s^*}{|s|^2}} \]

and so the integral converges at \(+\infty\) (remember that \( Re s^* = x > 0 \)). Similarly, for \( w \to -\infty \) the integrand goes like

\[ \sim \frac{\pi}{s^2} e^{i\pi s} \log \zeta(2 + iw) e^{\frac{2w^* s^*}{|s|^2}} \]

and so the integral converges at \(-\infty\). Being know that \( \zeta(2 + i w) \) has neither zeros nor poles for \( w \in \mathbb{R} \), we can say that (the Holomorphic Extension of) the integral in \( \sum - \int \) has no singularities for \( s \in \mathbb{C} \setminus \mathbb{R} \).

The inescapable conclusion is that all singularities of \( C(s) \) in the critical strip emerge from the difference between it and the corresponding integral, i.e. they are among the isolated points where \( F(s) = +\infty \).

4 Calculating the limiting function

Let recover the result (5):

\[ \left| \sum - \int \right| \leq \int_{M}^{+\infty} \frac{dp(t)}{dt} \left| \left| \frac{1}{p(t)(p(t)^s - 1) - \frac{sp(t)^{s-1} \log p(t)}{(p(t)^s - 1)^2}} \right| \right| \left| \int_{M}^{+\infty} \frac{dp(t)}{dt} I_t \left[ \right] - \int_{M}^{+\infty} \frac{dp(t)}{dt} [1 - I_t] \left[ \right] \right| \]

where \( I_t = 1 \) if \( \frac{dp(t)}{dt} \geq 0 \) and \( I_t = 0 \) otherwise. Use now \( dt \frac{dp(t)}{dt} = dp \) to achieve an
advantageous change of variable:

\[
\leq \int_{p_M}^{+\infty} dp \, I_t(p) \left| \left[ \quad \right] \right| - \int_{p_M}^{+\infty} dp \, [1 - I_t(p)] \left| \left[ \quad \right] \right|
\]

\[
\leq \int_{p_M}^{+\infty} dp \, I_t(p) \left| \left[ \quad \right] \right| + \int_{p_M}^{+\infty} dp \, [1 - I_t(p)] \left| \left[ \quad \right] \right|
\]

\[
\leq \int_{p_M}^{\infty} dp \left| \left[ \frac{1}{p(p^s - 1) - (p^s - 1)^2} \right] \right|
\]

\[
\leq \int_{p_M}^{\infty} dp \left| \frac{p^s - 1 - sp^{s-1} \log p}{p(p^s - 1)^2} \right|
\]

Now consider the following inequality:

\[
|p^s - 1|^2 = (p^s - 1)(p^{s^*} - 1) = p^{s+s^*} - p^s - p^{s^*} + 1
\]

\[
= p^{2x} - 2p^x \cos(y \log p) + 1
\]

\[
\geq p^{2x} - 2p^x + 1 = (p^x - 1)^2
\]

For \(x > 0\) we have also

\[
|p^s - 1 - sp^{s-1} \log p|^2 = (p^s - 1 - sp^{s-1} \log p)(p^{s^*} - 1 - s^{*}p^{s^{*-1}} \log p)
\]

\[
= p^{2x} - 2p^x \cos(y \log p) + 1 -
2xp^{2x-1} \log p + 2xp^{x-1} \log p \cos(y \log p) -
2yp^{x-1} \log p \sin(y \log p) + |s|^2 p^{2x-2} \log^2 p
\]

\[
\leq p^{2x} + 2p^x + 1 + 2xp^{2x-1} \log p + 2xp^{x-1} \log p +
2|y|p^{x-1} \log p + (x^2 + y^2)p^{2x-2} \log^2 p
\]
\[ \leq p^{2x} + 2p^x + 1 + 2xp^{2x-1} \log p + 2xp^{x-1} \log p + \\
+ 2\left| y \right| p^{x-1} \log p + (x^2 + y^2)p^{2x-2} \log^2 p + \\
+ 2x\left| y \right| p^{2x-2} \log^2 p + 2\left| y \right| p^{x-1} \log p = \\
= (p^x + 1 + (x + \left| y \right|)p^{x-1} \log p)^2 \]

Hence

\[ \left| \sum - \int \right| \leq \int_{p_M}^{\infty} dp \frac{p^x + 1 + (x + \left| y \right|)p^{x-1} \log p}{p(p^x - 1)^2} \left( \frac{\log(1-p^{-x})}{x} - \frac{(x + \left| y \right|) \log p((p^x - 1)\Phi(p^x, 1, -\frac{1}{2}) + x)}{x^2(p(p^x - 1)} \right) \bigg|_{p_M}^{\infty} \]

\[ \Phi(w, a, b) \] is the Lerch Transcendent\(^3\) which goes at \(+\infty\) in the first variable as \(w^{-1}\); in our case as \(p^{-x}\). As consequence, the contribute at \(+\infty\) is null. Finally

\[ \left| \sum - \int \right| \leq \frac{2}{x(p_M^x - 1)} + \frac{\log(1-p_M^{-x})}{x} + \frac{(x + \left| y \right|) \log p((p_M^x - 1)\Phi(p_M^x, 1, -\frac{1}{2}) + x)}{x^2p_M(p_M^x - 1)} \]

The Lerch Transcendent \(\Phi(w, a, b)\) has singularities (if \(a \in \mathbb{N} \setminus 0\)) only for \(b \in -\mathbb{N} \setminus 0\). In our case, \(a = 1\) and so we have singularities for

\[ x = \frac{1}{n} \quad \text{with} \quad n \in \mathbb{N} \setminus 0 \]

This means that the zeros of \(\zeta(s)\) in the strip \(0 < x < 1\) have to satisfy \(x = \frac{1}{n}\) for some \(n \in \mathbb{N} \setminus 0\). Moreover the functional equation

\[ \zeta(s) = 2^s \pi^{s-1} \sin \left( \frac{\pi s}{2} \right) \Gamma(1-s)\zeta(1-s) \]

reveals that if \(s_0\) is a non-trivial zero, then \(1 - s_0\) is a zero too. Hence we search

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\(^3\)\(\Phi(w, a, b)\) is usually defined as the holomorphic extension of \(\Phi(w, a, b) = \frac{1}{\Gamma(w)} \int_0^{\infty} \frac{e^{-t} t^{-w} dt}{(1 - e^{-t})^a} \) which works for \(\{Re b > 0 \land Re a > 0 \land \left| w \right| < 1\} \cup \{Re b > 0 \land Re a > 1 \land \left| w \right| = 1\}\).
for two integers $m, n$ such that

$$\frac{1}{m} = 1 - \frac{1}{n},$$

but the unique solution of this equation is $m = n = 2$. Consequently, all the non-trivial zeros of zeta must have real part equal to $1/2$. \textbf{CVD}